The Conjugate Space H^*

$H \to$ H.S. A continuous linear transformation from H into C is called a functional on H.

$\therefore$ Every functional on H is a continuous linear transformation i.e. cont. linear functional on H.

$B(H, C) \to$ set of all cont. linear functionals on H.

We denote $B(H,C)$ by H^*.

H^* is called conjugate space of H.

Thus if $f \in H^*$ then f is called a functional in H^* & f is a cont. linear functional on H.

We define norm of a functional 'f' as —

$$\|f\| = \sup\{|f(x)| : \|x\| \le 1\}$$

We can also define this norm as —

$$\|f\| = \sup\{|f(x)| : \|x\| = 1\},$$

if H is not a zero space.

Also $$\|f\| = \sup\left\{\frac{|f(x)|}{\|x\|} : x \neq 0\right\}$$

Thm → Let 'y' be a fixed vector in a H.S. H and let f_y be a scalar valued fn. on H defined by –

$$f_y(x) = (x, y) \quad \forall\, x \in H$$

Then f_y is a functional in H^*

i.e. f_y is a cont. linear functional on H.

Also, $\|y\| = \|f_y\|$

Pf → Given

$$f_y(x) = (x, y) \quad \forall\, x \in H \quad \text{---}(1)$$

Since (x, y) is a complex number $\forall\, x \in H$,

$\therefore$ f_y is a mapping from H into C.

Now to show that f_y is a functional on H, we have to show that f_y is linear & continuous.

f_y is linear: Let $x_1, x_2 \in H$

& let α, β be any scalars.

Then –

$$f_y(\alpha x_1 + \beta x_2) = (\alpha x_1 + \beta x_2, y); \text{ from (1)}$$

$$= \alpha (x_1, y) + \beta (x_2, y)$$

$$= \alpha f_y(x_1) + \beta f_y(x_2); \text{ from (1)}$$

$\therefore$ f_y is linear

f_y is continuous : we shall show that f_y is bounded as every bdd. fn. is cont. in H.

$\forall x \in H$, we have –

$$f_y(x) = (x, y)$$

$$\therefore |f_y(x)| = |(x, y)|$$

$$\leq \|x\| \|y\| \quad ; \text{ by Schwarz inequality} \rightarrow (2)$$

Let $\|y\| = K$. Then $K \geq 0$. (as y is a fixed vector)

$\therefore$ we have –

$$|f_y(x)| \leq K \|x\| \quad \forall x \in H.$$

$\Rightarrow f_y$ is bdd.

$\Rightarrow f_y$ is continuous.

Now, to show $\|y\| = \|f_y\|$

From (2), we have –

$$|f_y(x)| \leq \|x\| \|y\| \rightarrow (3)$$

By defn of norm, we have –

$$\|f_y\| = \sup\{|f_y(x)| : \|x\| \leq 1\} \rightarrow (4)$$

If $\|x\| \leq 1$, then $\|x\| \|y\| \leq \|y\|$

$\therefore$ from (3), we get –

$$|f_y(x)| \leq \|y\| \quad \forall x \text{ s.t. } \|x\| \leq 1$$

$$\Rightarrow \sup\{|f_y(x)| : \|x\| \leq 1\} \leq \|y\|$$

$$\Rightarrow \|f_y\| \leq \|y\| \rightarrow (5)$$

Now, if $y = 0$ then $\|y\| = 0$ and

$$f_y(x) = (x, y) = (x, 0) = 0 \quad \forall x \in H$$

$\therefore$ f_y is a zero fn. and
$\|f_y\| = 0$

i.e. if $y = 0$ then $\|y\| = 0 = \|f_y\|$.
So $\|y\| = \|f_y\|$ in this case.

Now let $y \neq 0$.

Then H is not a zero space,

$\therefore$ $\|f_y\| = \sup\{ |f_y(x)| : \|x\| = 1 \} \rightarrow (6)$

Since $y \neq 0$, $\therefore$ $\frac{y}{\|y\|}$ is a unit vector

as $\frac{\|y\|}{\|y\|} = 1$

$\therefore$ take $x = \frac{y}{\|y\|}$.

From (6), we get—

$$\|f_y\| \geq \left| f_y\left(\frac{y}{\|y\|}\right)\right|$$

$$= \left(\frac{y}{\|y\|}, y\right) \quad ; \because f_y(x) = (x, y)$$

$$= \frac{1}{\|y\|}(y, y)$$

$$= \frac{1}{\|y\|}\|y\|^2 = \|y\|$$

$\therefore$ $\|f_y\| \geq \|y\| \rightarrow (7)$

From (5) & (7), we get—

$\|f_y\| = \|y\|$

(Proved)